Generalizations of Hölder's and some related integral inequalities on fractal space

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Abstract: Based on the local fractional calculus, we establish some new generalizations of Hölder's inequality. By using it, some results on the generalized integral inequality in fractal space are investigated in detail.

Keywords: fractal space; local fractional calculus; reverse Hölder inequality

MSC2010: 28A80, 26D15

1 Introduction

While the renowned inequality of Hölder [1] is well celebrated for its beauty and its wide range of important applications to real and complex analysis, functional analysis, as well as many disciplines in applied mathematics. The purpose of this work is to establish some generalizations of Hölder inequality on local fractional calculus and other inequality based on it. Fractal calculus (also called local fractional calculus) has played an important role in not only mathematics but also in physics and engineers [2-15]. Local fractional derivative [6-8] were written in the form

$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}|_{x=x_0} = \lim\limits_{x\to x_0} \frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}$ for $0 < \alpha \leq 1$,

where

$$\Delta^\alpha(f(x)-f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x)-f(x_0)).$$

Local fractional integral of $f(x)$ [6-7,9] was written in the form

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_j, \ldots\}$, where for $j = 1, 2, \ldots, N-1 t_0 = a$ and $t_N = b$, $[t_j, t_{j+1}]$ is a partition of the interval $[a, b]$. Aims of this paper are to study the some new generalizations of Hölder's inequality and some results based on them.

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2 Some generalizations of Hölder inequality

In the section, we give some generalizations of Hölder inequality. In order to prove our results, we first review the Hölder inequality [15]:

Theorem 2.1 *.[15] Let $f(x)$, $g(x) \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)| \, (dx)^\alpha \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p \, (dx)^\alpha \right)^{1/p} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q \, (dx)^\alpha \right)^{1/q}, \tag{2.1}$$

equalities holding if and only if $f(x) = \lambda g(x)$, where λ is a constant.

Based on Theorem 2.1, we have the following important result.

Theorem 2.2 *Let $f(x)$, $g(x) \geq 0$, $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)| \, (dx)^\alpha \geq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p \, (dx)^\alpha \right)^{1/p} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q \, (dx)^\alpha \right)^{1/q}, \tag{2.2}$$

equalities holding if and only if $f(x) = \lambda g(x)$, where λ is a constant.

Proof. Set $c = 1/p$, then we have $q = -pd$, $d = c/(c-1)$. By (2.1), we obtain

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p \, (dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)|^p \, |g(x)|^{-p} \, (dx)^\alpha \\ &\leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)|^{pc} \, (dx)^\alpha \right)^{1/c} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^{-pd} \, (dx)^\alpha \right)^{1/d} \\ &= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)| \, (dx)^\alpha \right)^{1/c} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q \, (dx)^\alpha \right)^{1-p}. \end{aligned} \tag{2.3}$$

In (2.3), multiplying both sides by $\left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q \, (dx)^\alpha \right)^{p-1}$ yields

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p \, (dx)^\alpha \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q \, (dx)^\alpha \right)^{p-1} \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)| \, (dx)^\alpha \right)^p. \tag{2.4}$$

Using (2.4) implies that

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)| \, (dx)^\alpha \geq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p \, (dx)^\alpha \right)^{1/p} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q \, (dx)^\alpha \right)^{1/q}.$$

By Theorem 2.1 and Theorem 2.2, we give the following result.

Corollary 2.1 *Let $f_j(x) \geq 0$, $p_j \in \mathrm{R}$, $j = 1, 2, \ldots m$, $\sum\limits_{j=1}^{m} 1/p_j = 1$.*

(1) for $p_j > 1$, we have

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m} |f_j(x)| \, (dx)^\alpha \leq \prod_{j=1}^{m} (\int_a^b \frac{1}{\Gamma(1+\alpha)} |f_j(x)|^{p_j} \, (dx)^\alpha)^{1/p_j}. \tag{2.5}$$

(2) for $0 < p_1 < 1$, $p_j < 0$, $j = 2, \dots m$, we have

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m} |f_j(x)| \, (dx)^\alpha \geq \prod_{j=1}^{m} (\int_a^b \frac{1}{\Gamma(1+\alpha)} |f_j(x)|^{p_j} (dx)^\alpha)^{1/p_j}. \tag{2.6}$$

Proof. (1)We use induction on m. When $m = 2$, we are given p_1, $p_2 > 0$ with $1/p_1 + 1/p_2 = 1$. In particular, we have p_1, $p_2 > 1$ and so (2.5) is reduced to the Höder's inequality (2.1). Now suppose (2.5) holds for some integer $m \geq 2$. We claim that it also holds for $m + 1$. So let $p_1, p_2, \dots, p_{m+1} > 0$ be real numbers with $\sum\limits_{j=1}^{m+1} 1/p_j = 1$ and let $f_j(x) \geq 0$, $j = 1, 2, \dots m, m+1$. Note that, as above, we must have $p_i > 1$ for $j = 1, 2, \dots m, m+1$. In particular, we have

$$p_1 > 0, p_1/(p_1 - 1) > 0, 1/p_1 + (p_1 - 1)/p_1 = 1. \tag{2.7}$$

Thus by the Höder's inequality (2.1),

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m+1} |f_j(x)| \, (dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)| \prod_{j=2}^{m+1} |f_j(x)| \, (dx)^\alpha \\
&\leq (\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha)^{1/p_1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left(\prod_{j=2}^{m+1} |f_j(x)| \right)^{p_1/(p_1-1)} (dx)^\alpha \right)^{(p_1-1)/p_1} \\
&= (\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha)^{1/p_1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=2}^{m+1} |f_j(x)|^{p_1/(p_1-1)} (dx)^\alpha \right)^{(p_1-1)/p_1}.
\end{aligned} \tag{2.8}$$

Next, since

$$p_j(p_1 - 1)/p_1 > 0, \text{for} \quad j = 2, \dots m, m+1. \tag{2.9}$$

$$\sum_{j=2}^{m+1} p_1/p_j(p_1 - 1) = p_1(p_1 - 1) \sum_{j=2}^{m+1} 1/p_j = p_1(p_1 - 1)(1 - 1/p_1) = 1. \tag{2.10}$$

by induction hypothesis and (2.8), we arrive at

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m+1} |f_j(x)| (dx)^\alpha \leq (\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha)^{1/p_1} \\
&\times \left(\prod_{j=2}^{m+1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_j(x)|^{p_1/(p_1-1) \cdot p_j(p_1-1)/p_1} (dx)^\alpha \right)^{p_1/p_j(p_1-1)} \right)^{(p_1-1)/p_1} \\
&= (\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha)^{1/p_1} \prod_{j=2}^{m+1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_j(x)|^{p_j} (dx)^\alpha \right)^{1/p_j}.
\end{aligned} \tag{2.11}$$

Hence, we arrive at the result.

(2) Similar to the proof of (2.5), we use induction on m. Clearly when $m = 2$, equation (2.6) reduces to the Hölder's inequality (2.2). Now suppose that (2.6) holds for some integer $m \geq 2$. We claim that it also holds for $m + 1$. So let

$p_1, p_2, \ldots, p_m < 0$ and $p_{m+1} \in$ R be such that $\sum\limits_{j=1}^{m+1} 1/p_j = 1$ and let $f_j(x) \geq 0,\ j = 1, 2, \ldots m, m+1$. Note that $0 < p_{m+1} < 1$, since

$$p_1 > 0, 0 < p_1/(p_1 - 1) < 1, 1/p_1 + (p_1 - 1)/p_1 = 1. \tag{2.12}$$

by the Höder's inequality (2.2), we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m} |f_j(x)|(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)| \prod_{j=2}^{m} |f_j(x)|(dx)^\alpha \\
&\geq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha \right)^{1/p_1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left(\prod_{j=2}^{m} |f_j(x)| \right)^{p_1/(p_1-1)} (dx)^\alpha \right)^{(p_1-1)/p_1} \\
&= (\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha)^{1/p_1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=2}^{m} |f_j(x)|^{p_1/(p_1-1)} (dx)^\alpha \right)^{(p_1-1)/p_1}.
\end{aligned} \tag{2.13}$$

unless $\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha = 0$.

Now since

$$p_j(p_1 - 1)/p_1 < 0 \text{for} j = 2, \ldots m, p_{m+1}(p_1 - 1)/p_1 > 0 \tag{2.14}$$

and as in (2.10),

$$\sum_{j=2}^{m+1} p_1/p_j(p_1 - 1) = 1. \tag{2.15}$$

by induction hypothesis and (2.13), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m+1} |f_j(x)|(dx)^\alpha \geq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha \right)^{1/p_1} \\
&\times \left(\prod_{j=2}^{m+1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_j(x)|^{p_1/(p_1-1) \cdot p_j(p_1-1)/p_1} (dx)^\alpha \right)^{p_1/p_j(p_1-1)} \right)^{(p_1-1)/p_1} \\
&= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_1(x)|^{p_1} (dx)^\alpha \right)^{1/p_1} \prod_{j=2}^{m+1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_j(x)|^{p_j} (dx)^\alpha \right)^{1/p_j} \\
&= \prod_{j=1}^{m+1} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_j(x)|^{p_j} (dx)^\alpha \right)^{1/p_j}.
\end{aligned} \tag{2.16}$$

unless $\frac{1}{\Gamma(1+\alpha)} \int_a^b |f_j(x)|^{p_j} (dx)^\alpha = 0$ for some $j = 1, 2, \ldots m$.

3 Some related results

To set the stage, we recall Minkowski inequality [15]:

Theorem 3.1 *.[15] Let $f(x)$, $g(x) \geq 0$, $p > 1$, then*

$$\begin{aligned} &\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^p\,(dx)^\alpha\right)^{1/p} \\ &\leq \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p\,(dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q\,(dx)^\alpha\right)^{1/q}. \end{aligned} \tag{3.1}$$

equalities holding if and only if $f(x) = \lambda g(x)$, where λ is a constant.

Theorem 3.2 *.Let $f(x)$, $g(x) \geq 0$, $0 < p < 1$, then*

$$\begin{aligned} &\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^p\,(dx)^\alpha\right)^{1/p} \\ &\geq \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p\,(dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q\,(dx)^\alpha\right)^{1/q}. \end{aligned} \tag{3.2}$$

equalities holding if and only if $f(x) = \lambda g(x)$, where λ is a constant.

Proof. Let $M = \frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p\,(dx)^\alpha, N = \frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q\,(dx)^\alpha$ and

$$W = \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p\,(dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q\,(dx)^\alpha\right)^{1/q}.$$

By Hölder inequality, in view of $0 < p < 1$, we have

$$\begin{aligned} W &= \frac{1}{\Gamma(1+\alpha)}\int_a^b (|f(x)|^p\,M^{1/p-1} + |g(x)|^p\,N^{1/p-1})(dx)^\alpha \\ &\leq \frac{1}{\Gamma(1+\alpha)}\int_a^b (|f(x)+g(x)|^p\,(M^{1/p}+N^{1/p})^{1-p})(dx)^\alpha \\ &= W^{1-p}\frac{1}{\Gamma(1+\alpha)}\int_a^b (|f(x)+g(x)|^p)(dx)^\alpha. \end{aligned} \tag{3.3}$$

By inequality (3.3), we arrive to reverse Minkowski's inequality and the theorem is completely proved.

Theorem 3.3 *Let $f_j(x) \geq 0$, $j = 1, 2, \ldots m$,*

(1) for $p > 1$, we have

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p (dx)^\alpha\right)^{1/p} \leq \sum_{j=1}^m \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p\,(dx)^\alpha\right)^{1/p}. \tag{3.4}$$

(2) for $0 < p < 1$,we have

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p (dx)^\alpha\right)^{1/p} \geq \sum_{j=1}^m \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p\,(dx)^\alpha\right)^{1/p}. \tag{3.5}$$

Proof. (1) it follows from theorem 2.1 that

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\sum_{j=1}^m f_j(x)\right|^p(dx)^\alpha \le \frac{1}{\Gamma(1+\alpha)}\int_a^b|f_1(x)|\left|\sum_{j=1}^m f_j(x)\right|^{p-1}(dx)^\alpha\\
&+\cdots+\frac{1}{\Gamma(1+\alpha)}\int_a^b|f_m(x)|\left|\sum_{j=1}^m f_j(x)\right|^{p-1}(dx)^\alpha\\
&\le\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|f_1(x)|^p(dx)^\alpha\right)^{1/p}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\sum_{j=1}^m f_j(x)\right|^{q(p-1)}(dx)^\alpha\right)^{1/q} \qquad (3.6)\\
&+\cdots+\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|f_m(x)|^p(dx)^\alpha\right)^{1/p}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\sum_{j=1}^m f_j(x)\right|^{q(p-1)}(dx)^\alpha\right)^{1/q}\\
&=\sum_{j=1}^m\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|f_j(x)|^p(dx)^\alpha\right)^{1/p}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\sum_{j=1}^m f_j(x)\right|^{p}(dx)^\alpha\right)^{1/q}.
\end{aligned}
$$

in (3.6), multiplying both sides by $\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\sum\limits_{j=1}^m f_j(x)\right|^p(dx)^\alpha\right)^{1/q}$ yields (3.4).

(2) Similar to the proof of (3.2), we obtain (3.5).

Corollary 3.1 *Let* $f_j(x)\ge 0 j=1,2,\dots m$,

(1) for $p>1$, *we have*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\sum_{j=1}^m f_j(x)\right|^p(dx)^\alpha > \sum_{j=1}^m\frac{1}{\Gamma(1+\alpha)}\int_a^b|f_j(x)|^p(dx)^\alpha. \qquad (3.7)$$

(2) for $0<p<1$, *we have*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\sum_{j=1}^m f_j(x)\right|^p(dx)^\alpha < \sum_{j=1}^m\frac{1}{\Gamma(1+\alpha)}\int_a^b|f_j(x)|^p(dx)^\alpha \quad . \qquad (3.8)$$

Theorem 3.4 *Let* $f(x)$, $g(x)\ge 0$, $0<r<1<p$, *then*

$$
\begin{aligned}
&\left(\frac{\frac{1}{\Gamma(1+\alpha)}\int_a^b|f(x)+g(x)|^p(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_a^b|f(x)+g(x)|^r(dx)^\alpha}\right)^{1/(p-r)}\\
&\le\left(\frac{\frac{1}{\Gamma(1+\alpha)}\int_a^b|f(x)|^p(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_a^b|f(x)|^r(dx)^\alpha}\right)^{1/(p-r)}+\left(\frac{\frac{1}{\Gamma(1+\alpha)}\int_a^b|g(x)|^p(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_a^b|g(x)|^r(dx)^\alpha}\right)^{1/(p-r)}.
\end{aligned} \qquad (3.9)
$$

equalities holding if and only if $f(x)=\lambda g(x)$.

Proof. By Theorem 2.1 and Theorem 3.1, We have

$$
\begin{aligned}
&\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^p (dx)^\alpha\right)^{1/(p-r)}\\
&\leqslant \left(\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p (dx)^\alpha\right)^{1/p}+\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^p (dx)^\alpha\right)^{1/p}\right)^{p/(p-r)}\\
&=\left(\left(\frac{\int_a^b |f(x)|^p (dx)^\alpha}{\int_a^b |f(x)|^r (dx)^\alpha}\right)^{1/p}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/p}\right.\\
&\left.+\left(\frac{\int_a^b |g(x)|^p (dx)^\alpha}{\int_a^b |g(x)|^r (dx)^\alpha}\right)^{1/p}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/p}\right)^{p/(p-r)} \qquad (3.10)\\
&\leqslant \left(\left(\frac{\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p (dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha}\right)^{1/(p-r)}+\left(\frac{\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^p (dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha}\right)^{1/(p-r)}\right)\\
&\times\left(\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/r}+\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/r}\right)^{r/(p-r)}.
\end{aligned}
$$

Using reverse Minkowski inequality implies that

$$
\begin{aligned}
&\left(\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r\,(dx)^\alpha\right)^{1/r}+\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r\,(dx)^\alpha\right)^{1/r}\right)^{r}\\
&\le \frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^r\,(dx)^\alpha.
\end{aligned} \qquad (3.11)
$$

By (3.10) and (3.11), we get (3.9). Hence, the theorem is completely proved.

Corollary 3.2 *Let $f_j(x)\ge 0$, $0<r<1<p$,$j=1,2,\dots m$ then*

$$
\left(\frac{\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum\limits_{j=1}^m f_j(x)\right|^p (dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum\limits_{j=1}^m f_j(x)\right|^r (dx)^\alpha}\right)^{1/(p-r)} < \sum_{j=1}^m \left(\frac{\frac{1}{\Gamma(1+\alpha)}\int_a^b |f_j(x)|^p\,(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_a^b |f_j(x)|^r\,(dx)^\alpha}\right)^{1/(p-r)}. \qquad (3.12)
$$

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